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A geometrical interpretation of the coagulation equation

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Abstract. The coagulation–fragmentation equation describes geodesic motion in an infinite-dimensional space. This space has a symmetric affine connection but no metric in general. Some advantages of the new approach are indicated.

1. Introduction

The coagulation equation (CE) is a probabilistic, phenomenological equation pertaining to a system consisting of many separate masses or 'fragments' that can stick together pairwise. It describes the evolution of the mass spectrum. When the CE is generalised to include fragmentation of the colliding bodies, it becomes the so-called CFE or 'coagulation–fragmentation equation'. The CE, and to a much smaller extent the CFE, have been applied to such topics as rain-drop sizes (Melzak 1953), formation of the solar system (Safronov 1962, 1972), growth of dust grains (Hayashi and Nakagawa 1975), the initial stellar mass function (Nakano 1966, Silk and Takahashi 1979), and clustering in an expanding universe (Silk and White 1978).

At first sight, it appears that we know almost nothing about the generic solution of this highly nonlinear equation; strictly speaking, every solution depends both on the assumed form of the initial spectrum and on that of the coagulation coefficients, A_{ij}. But, in practice the situation is less bleak than this sounds. Numerical and analytical studies (Nakano 1966, Safronov 1972, Hayashi and Nakagawa 1975) demonstrate that the mass spectrum of the CE relaxes to a self-similar form after a few collision times. The limiting shape depends only on the form of the A_{ij}, not upon the initial spectrum. This property makes the CE extremely attractive for studying the formation of celestial systems because it replaces unknown initial conditions by known physics of coalescence.

My aim here is to sketch a new geometrical approach to the coagulation equation. In § 2, I rewrite the CFE as a geodesic equation for motion of a representative point in an infinite-dimensional space endowed with an affine connection. In § 3, I indicate some merits of this approach.

2. Interpreting coagulation as geodesic motion

Consider first, for simplicity, a system without fragmentation. Let $n(m, t)\,\mathrm{d}m$ be the volume density of fragments in the mass range $[m, m+\mathrm{d}m]$. The number of fragments in a given mass range increases when two smaller blobs of the right total mass coalesce;

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it will decrease whenever a blob of the specified mass agglomerates with any other piece (larger or smaller):

$$\frac{dn(m,t)}{dt}=\frac{1}{2}\int_0^m A(m',m-m')n(m',t)n(m-m',t)\,dm' - n(m,t)\int_0^\infty A(m,m')n(m',t)\,dm'. \tag{1}$$

(The factor $\frac{1}{2}$ comes in to compensate for double counting.)

The quantity $A(m, m')$ is called the coagulation coefficient and equals essentially σv, with σ the capture cross section and v the average relative speed between any two fragments.

Now to include the break-up resulting from collision. Let $w(m, m')$ be the probability that masses m, m' simply stick together upon colliding; then $1-w(m, m')$ gives the probability that they fragment after their mutual capture, perhaps breaking into many smaller pieces. Furthermore, let $n_1(m''/m, m')$ be the distribution function of the new pieces m'' produced thereby. By mass conservation,

$$\int_0^\infty m''n_1(m''/m,m')\,dm''=m+m' \tag{2}$$

$$n_1(m''/m,m')=0 \qquad \text{if } m''>m+m'. \tag{3}$$

We do not follow Safronov (1972) in putting $n_1=n_1(m''/m+m')$ because that amounts to assuming complete 'relaxation' during the collision, with all individuality of the parent blobs being lost. This would not be the case in general.

The generalised CE (with break-up) takes the form (Safronov 1972)

$$\begin{aligned}\frac{dn(m,t)}{dt}&=\frac{1}{2}\int_0^m w(m',m-m')A(m',m-m')n(m',t)\\&\quad\times n(m-m',t)\,dm'-n(m,t)\int_0^\infty A(m,m')n(m',t)\,dm'\\&\quad+\frac{1}{2}\int_m^\infty dm''\int_0^{m''}dm'\,n_1(m/m''-m',m')[1-w(m',m''-m')]\\&\quad\times A(m',m''-m')n(m',t)n(m''-m',t).\end{aligned} \tag{4}$$

One recovers the ordinary CE (without fragmentation) when $w=1$.

We now rewrite equation (4) as follows: (*a*) all limits of integration are extended to 0 or ∞ (this is permitted in view of (3) and the vanishing of $n(m, t)$ for negative values of mass); and (*b*) new dummy variables $m-m'\to m''$, $m''-m'\to m'''$ are introduced along with Dirac deltas and single integrals are replaced by double ones. Under (*a*) and (*b*), equation (4) changes to

$$\begin{aligned}\frac{dn(m,t)}{dt}&=\frac{1}{2}\int_0^\infty dm''\int_0^\infty dm'\,\{\delta(m-m'-m'')w(m',m'')\\&\quad-\delta(m-m')-\delta(m-m'')+n_1(m/m',m'')\\&\quad\times[1-w(m',m'')]\}A(m',m'')n(m',t)n(m'',t).\end{aligned} \tag{5}$$

It becomes more obvious that (5) is formally a geodesic-type equation if we translate it into the discrete case, where every mass (measured in some unit) takes an integer value $m \to j = 1, 2, 3 \ldots$, and $n(m, t) \to n^j(t)$. Equation (5) can then be written as

$$\mathrm{d}n^j/\mathrm{d}t + \Gamma^j_{kl} n^k n^l = 0 \tag{6}$$

where the affine connection has components

$$\Gamma^j_{kl} = \tfrac{1}{2}\{\delta^j_k + \delta^j_l - \delta^j_{(k+l)} w_{kl} - n_1(j/k, l)[1 - w_{kl}]\} A_{kl}. \tag{7*}$$

Note that we mark with an asterisk those equations where repeated indices are *not* summed over the set of all positive integers. Thus on both sides of (7*), j, k, l take unique values. It is perhaps not surprising that the connection is represented by a peculiar, non-covariant expression, because it is, as usual, not a tensor.

We see that (6) is of the familiar geodesic form

$$\mathrm{d}u^j/\mathrm{d}\tau + \tilde{\Gamma}^j_{kl} u^k u^l = 0 \tag{8}$$

where $u^j = \mathrm{d}x^j/\mathrm{d}\tau$ is the proper velocity, τ is the proper time (or some other affine parameter) and $\tilde{\Gamma}^j_{kl}$ is the affine connection.

The quantities $n_1(j/k, l)$, w_{kl} and A_{kl} are, by definition, symmetric in k and l since the order in which we consider the colliding fragments is immaterial. It follows that $\Gamma^j_{kl} = \Gamma^j_{lk}$, the same as in Riemannian geometry where Γ^j_{kl} equals the familiar Christoffel symbol.

The analogy of (7*) with (8) can be completed by introducing coordinates whose velocity components are the n^k. That is, we define N^k by $n^k = \mathrm{d}N^k/\mathrm{d}t$, or $N^k = \int n^k\, \mathrm{d}t$. A lexicon for the completed interpretation then runs as follows.

Coagulation Symbol (discrete; continuous)	Geometric Meaning
t	affine parameter
j; m	coordinate label
Γ^j_{kl}; $\Gamma(m/m', m'')$	connection coefficient
$n^j(t)$; $n(m, t)$	proper velocity
$N^j(t)$; $N(m, t)$	coordinate

3. Applications

I mention briefly some of the advantages of the present reformulation.

(i) The compact notation makes it easy to write down a Taylor expansion of $n^k(t)$ around $t = 0$, by taking successive differentiations of the CE;

(ii) If we define $M^p = \int_0^\infty m^p n(m, t)\, \mathrm{d}m$, then $\mathrm{d}M^p/\mathrm{d}t$ can be written by inspection. This includes the well known expressions for M^0 (total number of fragments) and M^1 (total mass density—conserved; see Melzak (1953));

(iii) One identifies two types of geometrical, conserved quantities: (a) $n^j\xi_j$, where $\xi_j = j$ is a Killing vector (i.e. $\xi_{j;k} + \xi_{k;j} = 0$ where the semicolon denotes the usual covariant derivative), and (b) $P \equiv g_{jk} n^j n^k$, where g_{jk} is a covariant 'metric' tensor satisfying

$$g_{ij;k} = g_{ij,k} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{li} = 0.$$

For the case $A_{ij} = \alpha^{-1}(\alpha + \beta i)(\alpha + \beta j)$, a particular solution of this equation is $g_{ij} \propto A_{ij} \exp[\Sigma_k (\alpha + \beta k) N^k]$. This can be superposed linearly with the general solution $g_{ij} \propto ij$ to obtain another allowable metric.

Our approach has economy and the suggestive power of geometry. But it remains to explore other facets of this geometric approach, e.g. (i) coordinate transformations and (ii) the Riemann curvature tensor $R^{i}_{\cdot jkl}$ (which can be defined solely in terms of the affine connection, without a metric; Adler *et al* (1965)). Then we shall know whether the formalism sketched here is something more than a mere curiosity.

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